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THE EFFECT OF VIRTUAL MASS ON THE BASIC EQUATIONS FOR UNSTEADY ONE-DIMENSIONAL HETEROGENEOUS FLOWS

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Abstract—A separated two-component flow model is presented which includes virtual mass forces coupling the momentum equations of the two components. It is shown that for physically realistic situations four real roots of the characteristic determinant can exist. These are associated with the acoustic propagation velocities and the flow velocities of the constitutive phases. Direct analytical solution of the full characteristic determinant is difficult. However, for low Mach number flows an acoustic propagation velocity is obtained which falls between the well-known true separated and homogeneous wave speeds, and compares favorably with experimental data for glass/water and air/water mixtures.

INTRODUCTION

One of the problems that has to be overcome when attempting to analyze the flow of two-component mixtures in pipes is the desire to properly describe the flow processes and at the same time have a set of equations that can be manipulated to give the necessary answers for design purposes. Recent studies have been directed mainly towards mixtures of water, air and water vapour with the principal applications being fault conditions in condenser cooling water systems and nuclear installations. The assumed physical models have ranged over a considerable spectrum. At one end are the homogeneous pseudo-fluid models having properties averaged over a cross section and a constant speed of pressure wave propagation, with the gas or vapour phase lumped in some cases at discrete intervals. Towards the other end have been those models which attempt to incorporate mass transfer, diffusion processes and variable wave speeds.

The mathematical description of these situations is a set of partial differential equations, the complexity of which is, naturally, linked to the complexity of the assumed physical model. The solution of these equations has, in general, been attempted by utilizing the method of characteristics although other schemes have been investigated to test their usefulness. Nevertheless, the method of characteristics has become one of the more widely used techniques among analysts of one-dimensional unsteady flows, partly because of its relative simplicity and the ease with which boundary conditions can be incorporated.

For flows which may be regarded as homogeneous and in thermal equilibrium, in which both components or phases are uniformly distributed, have the same velocities, rates of change of velocity and similar physical properties, the so-called three-equation system, namely a mixture momentum equation and separate equations for the conservation of mass of each phase, are usually found to be adequate (Wallis 1969). However, for other situations, especially where relative motion between the phases occurs, it is necessary that separate momentum equations be used for each of the components of the flow. Unfortunately, in developing the cross-sectionally averaged set of basic equations there is an unresolved conflict (Gidaspow 1974) over the question of whether they represent a "well-posed problem" and could, or should not, lead to some of the characteristics being imaginary.

Associated with the problem of complex characteristics is the fact that instabilities frequently occur during the numerical evaluation of the equations. These numerical problems have been shown (Bouré 1975) to be related to the modeling of the interaction between the phases. Two approaches are used to attempt to remedy this problem. In one an additional, somewhat hypothetical, viscosity term is introduced with a view to damping out high frequency disturbances in the numerical solution. The second, more recent approach is a series of attempts at developing a theoretical model which more accurately represents the interactions between the phases. These studies are primarily concerned with gas–liquid flows, some directed at bubbly flow and some at stratified flow.

In one study of particular interest, Drew *et al.* (1979) consider theoretically the virtual mass of a bubble in an accelerating two-phase flow, including the presence of other bubbles, and find that the virtual mass force f_{vm} can be expressed as

$$f_{vm} = C_{vm}\,\rho_c\,a_{vm} \qquad [1]$$

in which C_{vm} is the virtual mass per unit bubble volume (usually taken as ½ for a nondeformable single spherical bubble), and the virtual mass acceleration a_{vm} is given by

$$a_{vm} = \frac{\delta u_d}{\delta t} - \frac{\delta u_c}{\delta t} + u_d\frac{\delta u_d}{\delta x} - u_d\frac{\delta u_c}{\delta x} + \left[(\lambda - 2)(u_d - u_c)\frac{\delta u_d}{\delta x} + (1 - \lambda)(u_d - u_c)\frac{\delta u_c}{\delta x}\right]. \qquad [2]$$

The physical significance of the virtual mass is that it represents the sum of the actual mass of a discrete particle plus an added mass to make some allowance for the additional work done in accelerating fluid adjacent to the particle, as well as the particle itself.

In [2] u_d and u_c are the velocities of the discrete and continuous phases, respectively, at position x and time t.

The parameters C_{vm} and λ are both likely to be functions of the volume fraction α of the discrete phase. For steady flow, C_{vm} is taken as ½ for a sphere, but it will take different values for other shapes. The proximity of other discrete elements will affect C_{vm}, and may increase it (Mori *et al.* 1975) but little is known quantitatively about this. Similarly, the term λ requires further study. It is an arbitrary parameter introduced (Drew *et al.* 1979) in the development of [2] and must be found by experiment. However, Drew et al do demonstrate that it has limiting values of 2 and 0 when the void fraction α is zero and 1, respectively.

Using this formulation, i.e. [2], Lahey *et al.* (1980) found that the inclusion of added mass had very little effect on the accuracy of solution for a steady air–water flow, but that the numerical stability and efficiency (i.e. computer run time) was improved considerably. Consequently, they advocated it ought to be included in models of transient flows.

Hancox *et al.* (1980) also used [2] for a study of a blowdown situation. With $C_{vm} = 0.5$ and $\lambda = 0.5$ and 1 they encountered imaginary interfacial propagation velocities in stratified flow, before choked flow was established. By defining "mixture" properties of velocity and density they were able to force the basic equations into a hyperbolic set, though they comment that for mixed flows it is not clear whether the virtual mass effects were included in a self-consistent manner. This doubt arises because of the space-time averaging procedures used to yield a one-dimensional model, although it is recognized that distribution effects must be incorporated in the conservation equations. However, when doing this they believe that although hyperbolic equations are not essential, they are desirable because they have well-understood mathematical properties and well-established numerical solution procedures.

The objective of the present study is to investigate how the inclusion of virtual mass effects can influence the development of a "separated" flow model to be used for the transient flow of two-component mixtures in pipes. Inertial coupling between the components, or phases, will be included by means of a virtual mass force appearing in the momentum equations. The system is considered to be in thermal equilibrium, hence attention is restricted to flows such as low void fraction air/water mixtures or solid/liquid slurries.

CONSERVATION OF MASS AND OF LINEAR MOMENTUM

The generalized differential equations of conservation of mass and of linear momentum for one-dimensional unsteady two component flows may be expressed in terms of properties averaged over the flow cross section of area A. The conservation of mass of the discrete phase is given by

$$\frac{\delta}{\delta t}(\alpha\,\rho_d A) + \frac{\delta}{\delta x}(\alpha\,\rho_d u_d A) = 0 \qquad [3]$$

and for the continuous phase by

$$\frac{\delta}{\delta t}\left[(1-\alpha)\rho_c A\right] + \frac{\delta}{\delta x}\left[(1-\alpha)\,\rho_c u_c A\right] = 0. \qquad [4]$$

In these forms, it is further assumed that no mass transfer occurs between the phases, which have densities ρ_d and ρ_c for the discrete and continuous components and a volume fraction α defined as the volume of the discrete phase as a proportion of the whole.

The linear momentum equations may be defined in a similar fashion for the two components, i.e.

$$-\frac{\delta p}{\delta x} - (f_{Gd} + f_{Dd} + f_{vmd}) = \rho_d\left[\frac{\delta u_d}{\delta t} + u_d\frac{\delta u_d}{\delta x}\right], \qquad [5]$$

$$-\frac{\delta p}{\delta x} - (f_{Gc} + f_{Dc} + f_w - f_{vmc}) = \rho_c\left[\frac{\delta u_c}{\delta t} + u_c\frac{\delta u_c}{\delta x}\right]. \qquad [6]$$

In this formulation f_{Gd} and f_{Gc} represent weight components per unit volume of the constituents occupying the control volume. The terms f_{Dd} and f_{Dc} represent the drag force on the discrete elements and the reaction on the continuous phase, respectively, and wall friction f_w is assumed to apply to the continuous phase. The first term, $\delta p/\delta x$ is the previous gradient in the axial direction.

The virtual mass effect due to the influence of the discrete elements on the kinematic behaviour of the continuous phase is incorporated after the fashion of Drew *et al.* (1979):

$$f_{vmd} = \rho_c C_{vm}\; a_{vm} \qquad [7]$$

in which a_{vm} is given by [2]. The reaction on the continuous phase due to the virtual mass effect is denoted by f_{vmc} in [6]. For the present study, it is assumed that this will be equal in magnitude but opposite in direction to f_{vmd}.

Other assumptions concerning the flow are that the pressure is uniform throughout the cross section and the two phases are in thermal equilibrium with no heat transfer occurring across the pipe walls.

Evaluation of Propagation Velocities

Equations [3] to [6] may be expanded and rearranged in a matrix format:

$$\begin{bmatrix} a_1 & a_3 & 0 & 0 \\ b_3 & b_1 & 0 & 0 \\ 0 & 0 & d_1 & 1/\alpha \\ 0 & 0 & d_2 & \dfrac{-1}{1-\alpha} \end{bmatrix} \begin{Bmatrix} u_d \\ u_c \\ p \\ \alpha \end{Bmatrix}_t + \begin{bmatrix} a_2 & a_4 & 1 & 0 \\ b_4 & b_2 & 1 & 0 \\ 1 & 0 & u_d d_1 & u_d/\alpha \\ 0 & 1 & u_c d_2 & \dfrac{-u_c}{1-\alpha} \end{bmatrix} \begin{Bmatrix} u_d \\ u_c \\ p \\ \alpha \end{Bmatrix}_x = \begin{Bmatrix} f_d \\ f_c \\ 0 \\ 0 \end{Bmatrix} \qquad [8]$$

in which

$$a_1 = \rho_d + \rho_c C_{vm}; \qquad a_2 = \rho_d u_d + \rho_c C_{vm}[u_d + (\lambda - 2)(u_d - u_c)]$$

$$a_3 = -\rho_c C_{vm}; \qquad a_4 = \rho_c C_{vm}[(1-\lambda)(u_d - u_c) - u_d]$$

$$b_1 = \rho_c + \frac{\alpha}{1-\alpha}\rho_c C_{vm}; \quad b_2 = \rho_c u_c + \frac{\alpha}{1-\alpha}\rho_c C_{vm}[u_d - (1-\lambda)(u_d - u_c)]$$

$$b_3 = -\frac{\alpha}{1-\alpha}\rho_c C_{vm}; \qquad b_4 = \frac{-\alpha}{1-\alpha}\rho_c C_{vm}[(\lambda - 2)(u_d - u_c) + u_d]$$

$$d_1 = \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}p} + \frac{1}{K_d}; \qquad d_2 = \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}p} + \frac{1}{K_c}.$$

The d_1, d_2 terms allow for the variation of pipe cross-sectional area with pressure $(1/A \cdot \mathrm{d}A/\mathrm{d}p)$ and the compressibility of the discrete and continuous components, K_d and K_c, respectively.

Of the additional force terms incorporated only the virtual mass element is a function of derivatives, while the other parameters such as gravitational effects and viscous drag appear only in the right hand column vector of [8] as f_d and f_c. The eigenvalues ν, that is, propagation velocities, associated with [8] can be determined by evaluating the characteristic determinant:

$$\left\| \begin{matrix} a_2 - \nu a_1 & a_4 - \nu a_3 & 1 & 0 \\ b_4 - \nu b_3 & b_2 - \nu b_1 & 1 & 0 \\ 1 & 0 & d_1(u_d - \nu) & (u_d - \nu)/\alpha \\ 0 & 1 & d_2(u_c - \nu) & -(u_c - \nu)/(1-\alpha) \end{matrix} \right\| = 0. \qquad [9]$$

With the following definitions

$$y_d = u_d - \nu, \quad y_c = u_c - \nu, \qquad [10]$$

$$K = \frac{d_1}{1-\alpha} + \frac{d_2}{\alpha}. \qquad [11]$$

Equation [9], upon expansion and substitution of [10] and [11] becomes

$$
\begin{aligned}
y_d^2 y_c^2 K \rho_d \rho_c \Bigg[& -\left(\frac{\alpha}{1-\alpha}\right) C_{vm} \frac{y_d}{y_c} - \frac{1}{y_c}\left(\frac{\alpha}{1-\alpha}\right)(u_d - u_c)\, C_{vm}(\lambda - 1) \\
& - \frac{\rho_c}{\rho_d} \cdot \frac{1}{y_d} C_{vm}(\lambda - 2)(u_d - u_c) - \left(1 + \frac{\rho_c}{\rho_d} C_{vm}\right)\Bigg] \\
+ y_d y_c \rho_c & \left[\frac{y_d}{y_c}\left(\frac{1}{\alpha}\right)\left(\frac{\rho_d}{\rho_c} + \frac{C_{vm}}{1-\alpha}\right) + \frac{1}{y_c}\frac{C_{vm}}{\alpha}(\lambda - 2)(u_d - u_c)\left(\frac{1}{1-\alpha}\right)\right. \\
& \left. + \frac{y_c}{y_d}\left(\frac{1}{1-\alpha}\right) + \frac{C_{vm}}{(1-\alpha)^2} - \frac{1}{y_d}\frac{C_{vm}}{(1-\alpha)^2}(1 - \lambda)(u_d - u_c)\right] = 0 \\
& = F(\nu).
\end{aligned}
\qquad [12]
$$

In its present form, [12] cannot be solved analytically. However, by restricting its application to certain categories of flow, simplifications are achieved which can lead to solutions for the characteristic roots. These simplifications include omission of the virtual mass force and low Mach number approximation.

Omission of virtual mass force

With $C_{vm} = 0$, all virtual mass effects are negated and [15] reduces to

$$- K y_d^2 y_c^2 \rho_d \rho_c + y_d^2 \frac{\rho_d}{\alpha} + y_c^2 \frac{\rho_c}{1-\alpha} = 0. \qquad [13]$$

For subsonic flow, this relation has been shown by Lyczkowski *et al.* (1975) to possess two roots which are complex conjugates and two real roots. When the phase velocities are equal, or for single phase flow, the four roots become real. As mentioned in the introduction, it has been shown that the inclusion of virtual mass terms renders the system hyperbolic.

Low Mach number approximation

If one assumes that the component velocities are much lower than the acoustic velocities, then the convective acceleration terms in [8] can be considered negligible when compared to the time derivatives. Applying this concept to [12] one finds

$$y_d = y_c = \nu \qquad [14]$$

$$-\nu^4 K\left\{\rho_d\rho_c + \rho_d\rho_c C_{vm}\left[\frac{\alpha}{1-\alpha} + \frac{\rho_c}{\rho_d}\right]\right\} + \nu^2\left\{\frac{\rho_d}{\alpha} + \frac{\rho_c}{1-\alpha} + \rho_c C_{vm}\left[\frac{1}{\alpha(1-\alpha)^2}\right]\right\} = 0. \qquad [15]$$

The four roots of [15] are all real. Two of them are zero, giving vertical path lines on the x–t plane, along which propagate the interfacial waves. The remaining two, relating to propagation of acoustic waves, are given by

$$\nu = \pm\left\{\frac{\dfrac{\rho_d}{\alpha} + \dfrac{\rho_c}{1-\alpha} + \dfrac{\rho_c C_{vm}}{\alpha(1-\alpha)^2}}{\left[\dfrac{1}{(1-\alpha)K_d} + \dfrac{1}{\alpha K_c} + \dfrac{1}{\alpha(1-\alpha)}\dfrac{1}{A}\dfrac{dA}{dp}\right]\rho_d\rho_c\left[1 + C_{vm}\left(\dfrac{\alpha}{1-\alpha} + \dfrac{\rho_c}{\rho_d}\right)\right]}\right\}^{1/2} \qquad [16]$$

$$= \pm\, c_{vm},$$

where c_{vm} denotes the wave propagation speed including the virtual mass effects. Elimination of the virtual mass terms in [16] results in the so-called stratified wave speed relation (e.g. Wallis 1969), i.e.

$$c_s = \left\{\frac{\rho_d \rho_c}{\alpha\rho_c + (1-\alpha)\rho_d}\left[\frac{\alpha}{K_d} + \frac{1-\alpha}{K_c} + \frac{1}{A}\frac{dA}{dp}\right]\right\}^{-1/2}. \quad [17]$$

The term $\alpha\rho_c + (1-\alpha)\rho_d$ has been called a pseudo density. By contrast the homogeneous wave speed c_m (obtained by initially assuming that both components have the same velocity) is given by

$$c_m = \left\{[\alpha\rho_d + (1-\alpha)\rho_c]\left[\frac{\alpha}{K_d} + \frac{1-\alpha}{K_c} + \frac{1}{A}\frac{dA}{dp}\right]\right\}^{-1/2}. \quad [18]$$

Wallis (1969) has suggested that [18] should be applicable for a fine dispersion of particles in a liquid, whereas [17] should apply when true stratified flow exists between two components with no interfacial drag forces present.

PARAMETRIC STUDIES OF [12]

As previously indicated [12] cannot be solved analytically, but it is instructive to examine the influence of certain parameters, such as C_{vm} and λ for a typical system using realistic pipeline data. For example, in the following discussion of air–water flows in pipes of diameter D and wall thickness e it is assumed that:

$$u_c = 3\ \text{m/s}, \qquad u_d = 2\ \text{m/s},$$
$$\rho_c = 1000\ \text{kg/m}^3, \qquad \rho_d = 4.2949\ \text{kg/m}^3,$$
$$K_c = 2.05\ \text{GPa}, \qquad K_d = 0.355\ \text{MPa},$$
$$D/e = 20, \qquad E = 207\ \text{GPa}.$$

Here E denotes the elastic modulus of the pipe wall material.

It will be shown that four real roots will be obtained for many situations. Two of these will always be real and of a similar magnitude, and correspond to the propagation speed of the main transient pressure (i.e. water hammer) waves. These will be termed "primary roots." The other two roots, which are close to the individual phase speeds, will be termed "secondary" or "minor" roots for the purposes of discussion.

The influence of the virtual mass coefficient

Setting [12] = F (ν) and evaluating this for various values of ν over a range of virtual mass coefficients between 0 and 0.5 yields a set of curves as shown in figure 1. For this particular example $\lambda = 1$ and a void fraction of 0.1 was used. It is apparent that for $C_{vm} = 0$ there are two primary roots of [12] in the region of ± 290 m/s. As C_{vm} is increased the primary roots take on much lower values, eventually reaching ± 68 m/s approximately (they are not quite equal) when $C_{vm} = 0.5$.

An alternative way of illustrating the influence of C_{vm} is figure 2 which is a direct solution of [16] for the same conditions. The results from the two equations are very close. The upper curve on figure 2 corresponds both to [16] with $C_{vm} = 0$ and to [17] for stratified flow. The bottom curve is from [18] to which solutions of [12] and [16] become asymtotic as $C_{vm} \geq 0.5$. The experimental data superimposed on figure 2 is from Martin & Padmanabhan (1979) for transient propagation speeds in air–water slug flow.

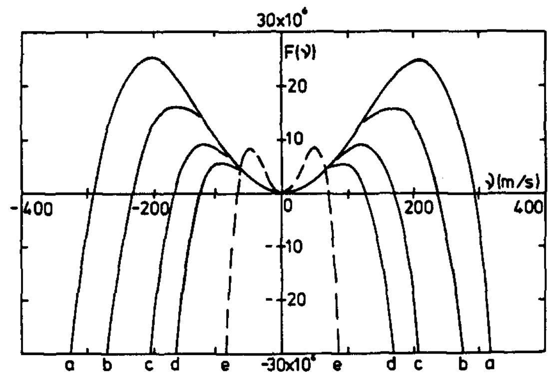

Figure 1. Effect of varying the added mass coefficient on F(ν), i.e. [12].

Curve	*a*	*b*	*c*	*d*	*e*
C_{vm}	0	0.0025	0.01	0.025	0.5

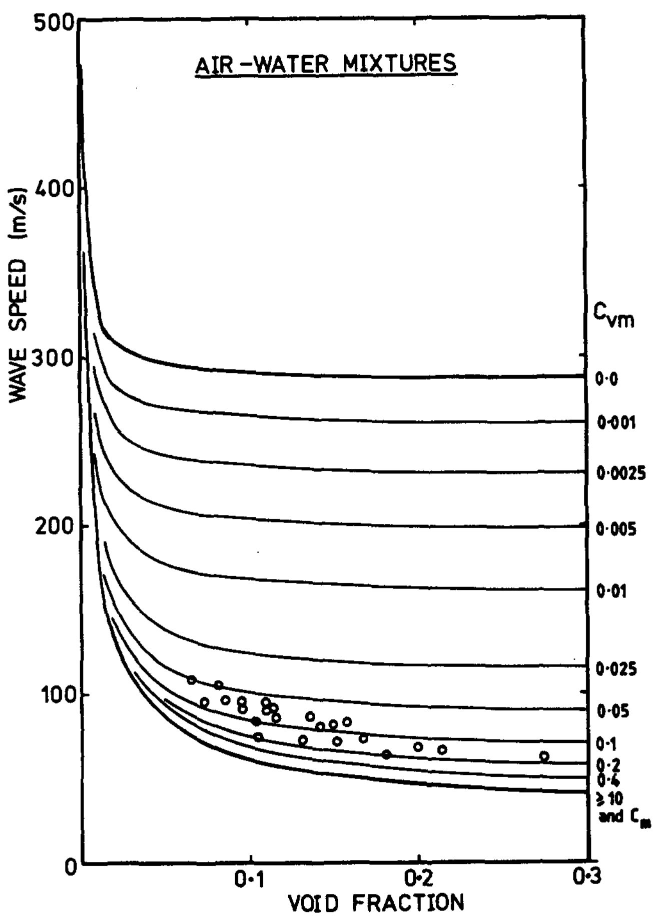

Figure 2. Speed of sound vs void fraction in air–water mixtures with different added mass coefficients. Experimental data are from Martin and Padmanabhan (1979).

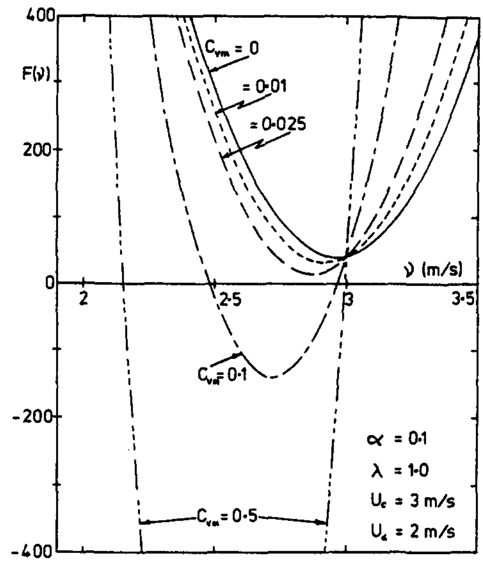

Figure 3. Effect of the added mass coefficient on the minor roots of [12].

In figure 1 all of the curves appear to fall towards zero as ν tends to zero. Closer examination, as shown in figure 3 shows that if the virtual mass coefficient exceeds approximately 0.03, as it would do in practical situations, there will be two real roots to [12]. Another interesting feature of this set of curves is that they all intersect at $\nu = 3$ m/s which was the assumed speed of the continuous phase. This feature is also true at different phase velocities and is irrespective of which phase has the greater speed.

The influence of the parameter λ

Drew *et al.* (1979) introduce λ into their development of a relationship for virtual mass acceleration, for which the acceleration is defined as [2]. The value of λ is expected to be a function of void fraction α at least, and from a consideration of limiting cases deduce that in the low and high void regimes:

$$\lim_{\alpha \to 0} \lambda(\alpha) = 2, \qquad [19]$$

$$\lim_{\alpha \to 1} \lambda(\alpha) = 0. \qquad [20]$$

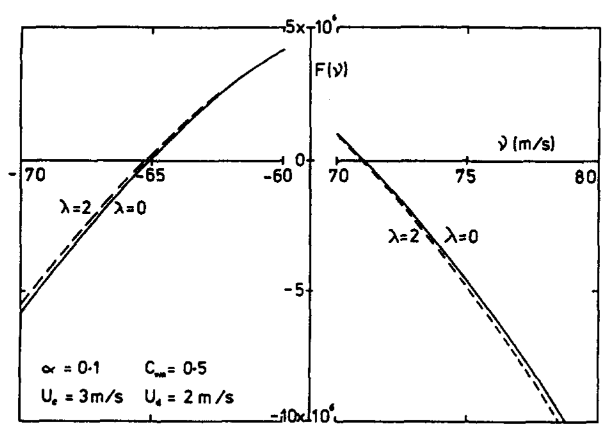

Figure 4. Effect of λ on the primary roots of [15] for a void fraction of 0.1.

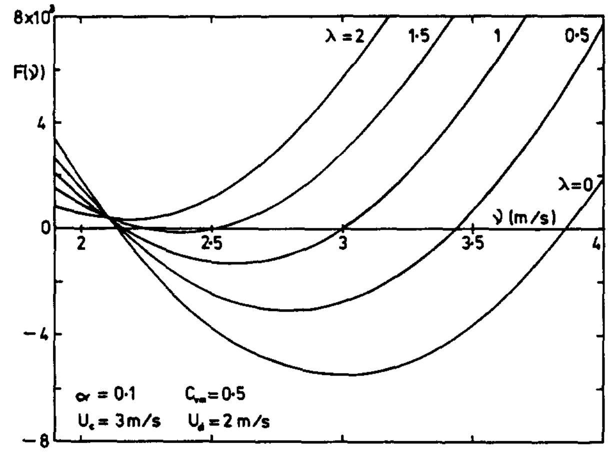

Figure 5. Effect of λ on the minor roots of [12] for a void fraction of 0.05.

Intermediate values of λ (α) are not clear. However, using realistic pipeline data as in the previous section the effect on the roots of [12] has been examined.

Figure 4 is typical of several that illustrate that the influence of λ on the numerical value of the primary roots is minimal. Only the two extreme values of 0 and 2 are plotted as the curves are so close. Other data for intermediate values of λ fall between these extremes. The effect on the secondary roots is more dramatic as shown in figure 5. Similar diagrams can be prepared for other void fractions. An extensive collection of this data has been combined in figures 6 and 7 which are graphs of the loci of the secondary roots of [12]. Two figures are used to display this data in the interests of clarity.

From figure 6 it is noted that for void fractions up to 0.2 there are two real secondary roots to [12] provided λ is less than about 1.6. As α decreases towards zero the limiting value of λ for two real secondary roots to exist moves towards 2, the value deduced by Drew *et al.* (1979) in [19]. Irrespective of what value of void fraction occurs, for λ = 1, one of the roots is always the velocity of the continuous phase, i.e. $\nu/u_c = 1$. The other root is slightly greater than the velocity of the discrete phase (0.667 in figure 6), but approaching it as the void fraction falls. At higher void fractions, above approximately 0.25 in this case, two real secondary roots occur for all values of λ. Again, for λ = 1, one root is the velocity of the continuous phase, whereas the other root is a slowly varying function of λ.

In general therefore, of the four possible real roots (assuming $\lambda \lesssim 1.6$) the one most

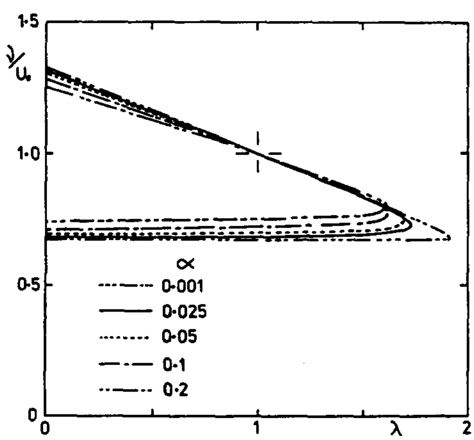

Figure 6. Loci of the minor roots of [12] vs λ for void fractions in the range 0.001–0.2.

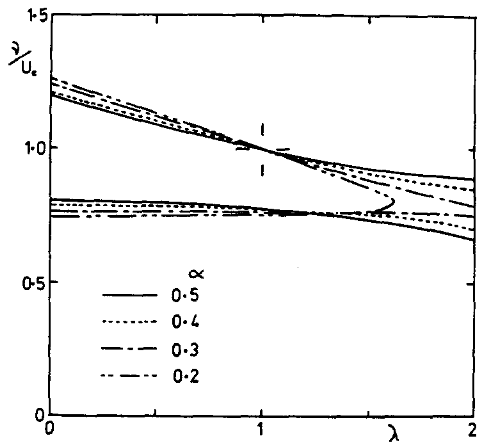

Figure 7. Loci of the minor roots of [12] vs λ for void fractions in the range 0.2–0.5.

significantly influenced is that corresponding to the speed of the continuous phase when $\lambda = 1$.

AN APPLICATION TO SOLID–LIQUID MIXTURES

Consider a mixture of glass beads in water with the following properties:

$$\rho_d = 2475\ \mathrm{kg/m^3}, \quad K_d = 39.62\ \mathrm{GPa},$$
$$\rho_c = 1000\ \mathrm{kg/m^3}, \quad K_c = 2.05\ \mathrm{GPa},$$
$$C_{vm} = 0.40, \quad c_0 = 1.432\ \mathrm{km/s}.$$

In the above $c_0 = \sqrt{K_c/\rho_c}$, the acoustic velocity in the continuous medium. Dimensionless plots of c_{vm}/c_0, c_s/c_0 and c_m/c_0 as functions of α are shown in figure 8. The acoustic speed which includes virtual mass effects falls between the stratified speed and the homogeneous speed. Figure 9 shows comparisons with experimental data for a similar mixture (Thorley 1978).

Comparing figure 2 with figures 8 and 9 it is noted that the deviation between homogeneous and separated flow model predictions of wave propagation speeds is far greater with air–water mixtures than with glass-bead and water mixtures. For the solid–liquid

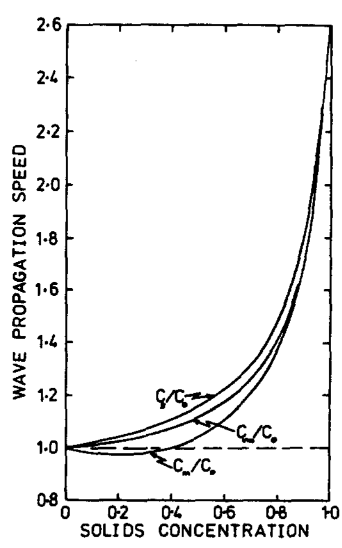

Figure 8. Theoretical wave speeds vs solids concentration for a glass bead/water mixture.

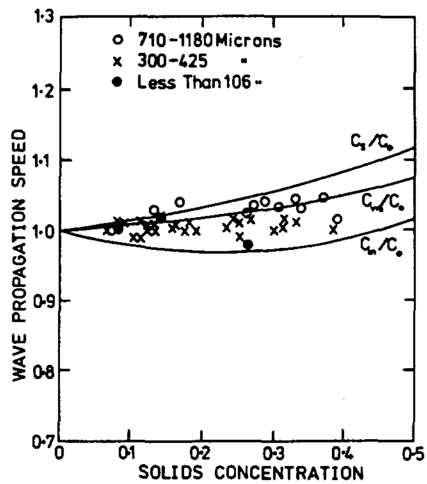

Figure 9. Comparison of experimental and theoretical wave speeds for a glass bead/water mixture.

mixtures the influence of varying the virtual mass coefficient is similar to that previously demonstrated for the air–water mixtures. With the virtual mass coefficient set equal to zero the solution of [16] is identical to that for [17]. Increasing C_{vm} in steps towards 0.5 brings the solution of [16] towards that for [18], approaching it asymptotically as C_{vm} is increased, but never crossing it.

DISCUSSION AND CONCLUSIONS

Assuming a separated flow model with thermal equilibrium, the four equation system (two conservation of mass and two conservation of momentum) can be formulated to include virtual mass forces due to the influence of the discrete phase on the kinematic behaviour of the continuous phase. In effect, this produces a coupling between the two momentum equations in terms of the derivatives of u_d and u_c—see [8]. The eigenvalues, i.e. propagation velocities, associated with this system of equations cannot be solved analytically. However, in contrast to the recent conclusions of Rath (1981), it is apparant that four real roots can exist, two primary roots and two secondary roots.

Using an estimated, but physically realistic virtual mass coefficient the two primary roots are essentially the same as the propagation speeds predicted by the well-known homogeneous model [18]. Two real secondary roots have also been shown to exist provided that both the virtual mass coefficient C_{vm} and the parameter λ are assigned values that can be justified on physical grounds. This qualification is not very restrictive as real roots can still be predicted using extreme values of C_{vm} and λ. Numerically, the two secondary roots are close to the velocities of the two separate phases and for $\lambda = 1$, one root is the actual velocity of the continuous phase.

It therefore appears that the addition of virtual mass provides a more accurate and generalised expression for the accoustic propagation velocity, one which may be applicable to a variety of heterogeneous flows. Uncertainties in the model include the precise form of the virtual mass term, [7] and the arbitrary omission of space-time distribution effects (Hancox *et al.* 1980). Wallis (1978) has remarked that although inertial coupling exists in most two-phase flows it is unlikely that a universal expression is valid for all flow regimes. In addition, he suggests that the coupling is significant only when the velocities are changing rapidly.

REFERENCES

BOURÉ, J. 1975 On a unified presentation of the non-equilibrium two-phase flow models, *ASME Symposium Volume Non-equilibrium two-phase flows.*

DREW, D. A., CHENG, L. & LAHEY, R. T. 1979 The analysis of virtual mass effects in two-phase flow. *Int. J. Multiphase Flow,* **5**, 233–242.

GIDASPOW, D. 1974 Modelling of two-phase flow. Round Table Discussion, Procs. 5th International Heat Transfer Conference, pp. 163–170.

HANCOX, W. T., FERCH, R. L., LIU, W. S. & NIEMAN, R. E. 1980 One-dimensional models for transient gas–liquid flows in ducts. *Int. J. Multiphase Flow,* **6**, 25–40.

HINZE, J. O. 1961 Momentum and mechanical energy balance equations for a flowing homogeneous suspension with slip between the two phases. *Appl. Sci. Res. Sect. A,* Vol. II.

LAHEY, R. T. Jr., CHENG, L. Y., DREW, D. A. & FLAHERTY, J. E. 1980 The effect of virtual mass on the numerical stability of accelerating two-phase flows. *Int. J. Multiphase Flow* **6**, 281–294.

LYCZKOWSKI, R. W., GIDASPOW, D., SOLBRIG, C. W. & HUGHES, E. D. 1975 Characteristics and stability analyses of transient one-dimensional two-phase flow equations and their finite difference approximations. ASME Paper 75-WA/HT-23.

MARTIN, C. S. & PADMANABHAN, M. 1979 Pressure pulse propagation in two-component slug flow. *J. Fluids Eng.* **101,** No. 1, 44–52.

MORI, Y., HIJIKATA, K. & KOMINE, A. 1975 Propagation of pressure waves in two-phase flow. *Int. J. Multiphase Flow,* **2**, 139–152.

RATH, H. J. 1981 Non-linear propagation of pressure waves in elastic tubes containing bubbly air–water mixtures. Procs. 5th International Symposium on Water Column Separation, IAHR, Obernach.

THORLEY, A. R. D. 1978 Experimental and theoretical studies of pressure surge phenomena in slurries. Research Report, Institute for Mining and Minerals Research, University of Kentucky, Lexington.

WALLIS, G. B. 1969 One-dimensional two-phase flow. McGraw Hill, New York.

WALLIS, G. B. 1978 Discussion of the paper on inertial coupling in dynamic equations of components in a mixture. *Int. J. Multiphase Flow* **4,** 585–586.